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AN EQUATION SET FOR NON-EQUILIBRIUM TWO PHASE FLOW, AND AN ANALYSIS OF SOME ASPECTS OF CHOKING, ACOUSTIC PROPAGATION, AND LOSSES IN LOW PRESSURE WET STEAM

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Abstract—An equation set for multidimensional, time variant, inviscid flow of a condensing vapour is presented. The equations include the effects of relative motion between the primary gas phase and the suspended liquid droplets. They have been formulated with steam turbine applications in mind but are also relevant to problems of gas-particle and liquid bubble flow.

It is shown that the critical velocity in one dimensional choking of low pressure wet steam is identical with the "frozen" speed of acoustic propagation, and the variation of choking mass flow with respect to equilibrium based calculations is described. Results obtained with two different models of droplet growth are compared, and simple formulae for calculating limiting values of choking flow are given. A generalised loss coefficient including the effects of thermodynamic and kinematic non-equilibrium is introduced.

1. INTRODUCTION

One class of multiphase flow problems which lend themselves readily to mathematical description are those involving small droplets or bubbles suspended within a primary gaseous or liquid phase. Formulating a mathematical model to deal with such flows entails setting down the equations of motion and energy conservation for the separate phases with correct accounting of the transfer of mass, momentum, and energy between the phases. This matter has presented some difficulties in the past, for example at a 1967 Euratom conference, Wundt remarked "It is a staggering fact that there is no distinctness in the bibliography on such a relatively simple matter as is the question about the necessary and sufficient number of equations and variables to describe multiphase gas-liquid flow". Similarly Deich & Seleznev (1968) observed "The fundamental equations of motion persist as one of many outstanding problems in the mechanics of two phase media".

Equation sets have appeared frequently in the literature, e.g. Gyarmathy (1962), Deich (1968), Kirillov (1970), Moore (1976), but restrictive assumptions appropriate to particular problems have usually been made, for example relative motion between the phases is often neglected, droplet temperature dynamics simplified and the equations given in steady one dimensional form. The papers concerned with a more general view have tended not to explore questions surrounding the choice of interphase transfer terms. It is the intention here to present an equation set which is sufficiently general to form a basis for the solution of a wide range of multiphase flow problems, and yet is specific enough regarding the transfer terms for a gas/droplet system to be of immediate use to engineers concerned with non-equilibrium calculations in wet steam which arise in turbines, heat exchangers, etc. The application of the equations is illustrated through the fundamental problems of choking, acoustic propagation, and loss, where restrictive assumptions can have significant effects.

The mathematical model presented describes a continuous vapour phase containing a distribution of liquid droplets of varying sizes and velocities which may be in the process of nucleation, condensation or evaporation. With simple modifications of the terms the equations for a liquid bubble system or chemically reacting gas particle flow can be obtained. It is impractical to include the detail of individual droplet flow fields in the model for this would require the solution of a non-steady multiply connected flow problem. One must therefore seek a macroscopic representation of the nett flux contributions of each phase in the continuity, momentum and energy equations. Following the approach of Kirillov *et al.* (1970), and Deich &

Fillipov (1968) "smoothing operations" are postulated to deal with discontinuities at the phase boundaries, and the droplet properties (e.g. temperature and velocity) are treated as continuous and differentiable functions of space and time. The modelling assumptions are realistic so long as the gas properties in the far field of a droplet do not change significantly over distances of the order of a droplet diameter. The droplets are taken to be incompressible and spherical, viscous forces at the external boundaries of the gas phase are not considered, but the droplet drag and heat interactions are included.

The basis of the flow model

The liquid phase is divided into droplet groups designated by the subscript i and distinguished by the size of the member droplets at some specified position in the flow field. The droplets once labelled remained in the same group throughout their flow history.

Let δv be the volume of an element of gas-droplet mixture comprising $\delta v''$ of gas, and $\delta v'$ of liquid

then

$$\delta v = \delta v'' + \delta v'$$

where

$$\delta v' = \sum_i \delta v_i$$

and δv_i is the volume occupied by the ith droplet group. A similar relationship holds for the component masses δm, $\delta m''$ and $\delta m'$, and we also have

$$\delta m = \rho \delta v \quad \delta m'' = \rho'' \delta v'' \quad \delta m_i = \rho_i \delta v_i$$

where ρ, ρ'', and ρ_i are the densities of the mixture, gas, and liquid making up the ith droplet group.

The droplet group mass fraction

$$y_i = \frac{\delta m_i}{\delta m} = \frac{\rho_i \delta v_i}{\rho \delta v} = n_i m_i.$$

The total wetness fraction

$$y = \sum_i \frac{\delta m_i}{\delta m} = \sum_i y_i$$

and the gas fraction

$$(1 - y) = \frac{\delta m''}{\delta m} = \frac{\rho'' \delta v''}{\rho \delta v}. \qquad [1.1]$$

Consider the mass flows of the various phases through a small surface $\delta \mathbf{s}$. A surface threading droplets of the ith group sweeps out a volume $(\mathbf{u}_i \cdot \delta \mathbf{s})\delta t$ in a time increment δt. The mass of the ith phase contained is $\rho y_i(\mathbf{u}_i \cdot \delta \mathbf{s})\delta t$ so we have the flux components $\rho y_i(\mathbf{u}_i \cdot \delta \mathbf{s})$ for the ith group and $\rho(1 - y)(\mathbf{c} \cdot \delta \mathbf{s})$ for the gas. If we imagine the total surface to be divided into sub-surfaces passing each phase alone

$$\delta \mathbf{s} = \delta \mathbf{s}'' + \sum_i \delta \mathbf{s}_i$$

then the continuity relationships require

$$\delta s_i = \frac{\rho y_i}{\rho_i}\delta s \quad \text{and} \quad \delta s'' = \rho\frac{(1-y)}{\rho''}\delta s. \qquad [1.2]$$

Following Kirillov *et al.* (1970) the hydrostatic pressure is assumed to act on the appropriate sub-surface when we formulate equations of motion for the individual phases. This procedure preserves the correct additive relationships between component and overall mixture equations. From [1.1]

$$\frac{1}{\rho} = \frac{(1-y)}{\rho''} + \sum_i \frac{y_i}{\rho_i}. \qquad [1.3]$$

If all the liquid droplets are of equal density ρ', and low pressure steam conditions prevail, then $\rho' \gg \rho''$ and $\rho \simeq \rho''/(1-y)$. With these assumptions the droplets are reduced to point sources or sinks of heat and matter distributed throughout the gas, but having, possibly, a relative motion with respect to it.

2. THE FLOW EQUATIONS

The flow equations for each phase may be derived, e.g. Jackson (1976), by considering the motion of a surface S threading elements of the phase and enclosing a variable volume V. Mass, momentum and energy balances for the volume lead initially to integral equations. The differential forms are then obtained via Gauss's theorem.

Continuity equations

For the gas phase the integral equation reads

$$\int_v \frac{\partial}{\partial t}(\rho(1-y))\,\mathrm{d}v + \int_s \rho(1-y)\mathbf{c}\cdot\mathrm{d}\mathbf{s} = -\sum_i \int_v n_i\dot{m}_i\rho\,\mathrm{d}v \qquad [2.1]$$

and the differential form is

$$\frac{\partial}{\partial t}(\rho(1-y)) + \nabla\cdot(\rho(1-y)\mathbf{c}) = -\rho\sum_i n_i\dot{m}_i. \qquad [2.2]$$

The term $n_i\dot{m}_i$, which represents the total transfer of mass from the gas to droplets of the ith group in unit mass of mixture, is replaced by $J_i m_i$ if the droplet group is in the process of nucleating.

For a droplet group integral and differential forms are:

$$\int_v \frac{\partial}{\partial t}(\rho y_i)\,\mathrm{d}v + \int_s \rho y_i\mathbf{u}_i\cdot\mathrm{d}\mathbf{s} = \int_v n_i\dot{m}_i\rho\,\mathrm{d}v \qquad [2.3]$$

$$\frac{\partial}{\partial t}(\rho y_i) + \nabla\cdot(\rho y_i\mathbf{u}_i) = \rho n_i\dot{m}_i \qquad [2.4]$$

or the equation for conservation of droplet number

$$\frac{\partial}{\partial t}(\rho n_i) + \nabla(\rho n_i\mathbf{u}_i) = \rho J_i$$

may be used in place of [2.4] where J_i denotes the nucleation rate. Summing the individual phase equations gives the overall continuity expressions for the mixture and the r.h.s. source terms vanish. Corresponding summations in the momentum and energy equations also remove weightings in pressure terms.

Momentum equations

The momentum transferred between the phases is $\dot{m}\mathbf{c}_G$ where $\mathbf{c}_G$ is the velocity of the phase from which the transfer proceeds, i.e. for condensation

$$\dot{m}_i > 0 \quad \mathbf{c}_G = \mathbf{c}$$

and for evaporation

$$\dot{m}_i < 0 \quad \mathbf{c}_G = \mathbf{u}_i.$$

For the gas phase:

$$\frac{\partial}{\partial t}\int_v \rho(1-y)\mathbf{c}\,\mathrm{d}v + \int_s \rho(1-y)\mathbf{c}(\mathbf{c}\cdot\mathrm{d}\mathbf{s}) + \sum_i \int_v n_i\dot{m}_i\,\mathbf{c}_G\,\rho\,\mathrm{d}v = -\int_s \frac{\rho(1-y)}{\rho''}p\,\mathrm{d}\mathbf{s} - \sum_i \int_v n_i\mathbf{D}_i\rho\,\mathrm{d}v. \tag{2.5}$$

Applying Gauss's theorem, shrinking the arbitrary volume to zero, and subtracting the product of the continuity equation [2.2] with $\mathbf{c}$, gives the differential form

$$\rho(1-y)\frac{D\mathbf{c}}{Dt} = -\nabla\left(\frac{\rho(1-y)}{\rho''}p\right) - \sum_i \rho n_i\mathbf{D}_i + \sum_i \rho n_i\dot{m}_i(\mathbf{c}-\mathbf{c}_G) \tag{2.6}$$

where $D/Dt = \partial/\partial t + \mathbf{c}\cdot\nabla$ denotes differentiation following the motion of the gas phase i.e. a material derivative.

For a droplet group:

$$\frac{\partial}{\partial t}\int_v \rho y_i\mathbf{u}_i\,\mathrm{d}v + \int_s \rho y_i\mathbf{u}_i(\mathbf{u}_i\cdot\mathrm{d}\mathbf{s}) - \int_v n_i\dot{m}_i\mathbf{c}_G\rho\,\mathrm{d}v = -\int_s \frac{\rho y_i}{\rho_i}p\,\mathrm{d}\mathbf{s} + \int_v n\mathbf{D}_i\rho\,\mathrm{d}v. \tag{2.7}$$

Applying the same procedures as for the gas phase gives

$$\rho y_i\frac{D\mathbf{u}_i}{Dt_i} = -\nabla\left(\frac{\rho y_i}{\rho_i}p\right) + \rho n_i\mathbf{D}_i + \rho n_i\dot{m}_i(\mathbf{c}_G-\mathbf{u}_i). \tag{2.8}$$

Energy equations

The rate of energy transfer between each droplet and the gas phase is $\dot{m}(\hat{e}_G + c_G^2/2)$ where $\hat{e}_G$ is the internal energy of the saturated vapour at the droplet surface as distinct from e'' the energy of the gas in the far field which may be either superheated or supercooled. Konorski (1976) takes account of distributions of temperature both within the droplets and in the gas field and gives expressions for the mass averaged energies of the phases which depend on the mean distance between droplets. No such corrections are made here.

Equating the energy change for a moving volume of gas with the total heat and work transfer gives

$$\sum_i \int_v n_i\dot{q}_i\rho\,\mathrm{d}v - \sum_i \int_v n_i\mathbf{D}_i\cdot\mathbf{u}_i\rho\,\mathrm{d}v - \int_s \rho(1-y)\frac{p}{\rho''}\mathbf{c}\cdot\mathrm{d}\mathbf{s} + \sum_i \int_v n_i\dot{m}_i\hat{p}_s\left(\frac{1}{\rho_i}-\frac{1}{\rho_G}\right)\rho\,\mathrm{d}v$$
$$= \frac{D}{Dt}\int_v \rho(1-y)\left(e''+\frac{c^2}{2}\right)\mathrm{d}v + \sum_i \int_v n_i\dot{m}_i\left(\hat{e}_G+\frac{c_G^2}{2}\right)\rho\,\mathrm{d}v$$

where $\dot{q}_i$ and $\mathbf{D}_i$ are the heat transfer rate and drag of an individual droplet respectively. The term Dv_i/Dt has been dropped since the droplets are assumed incompressible but it would be of importance in a liquid-bubble system.

Now for a scaler g

$$\frac{D}{Dt}\int_v g\,\mathrm{d}v = \frac{\partial}{\partial t}\int_v g\,\mathrm{d}v + \int_v \nabla\cdot(g\mathbf{c})\,\mathrm{d}v.$$

So applying Gauss's theorem and regrouping to introduce the enthalpies

$$\hat{h}_G = \hat{e}_G + \frac{\hat{p}_s}{\rho_G} \qquad h'' = e'' + \frac{p}{\rho''}$$

we obtain the integral equation

$$\sum_i \int_v n_i\dot{q}_i\rho\,\mathrm{d}v - \sum_i \int_v n_i\mathbf{D}_i\cdot\mathbf{u}_i\rho\,\mathrm{d}v = \frac{\partial}{\partial t}\int_v \rho(1-y)\left(e'' + \frac{c^2}{2}\right)\mathrm{d}v$$
$$+ \int_s \rho(1-y)\left(h'' + \frac{c^2}{2}\right)(\mathbf{c}\cdot\mathrm{d}\mathbf{s}) + \sum_i \int_v n_i\dot{m}_i\left(\hat{h}_G + \frac{c_G^2}{2}\right)\rho\,\mathrm{d}v. \qquad [2.9]$$

Subtracting the product of the continuity equation [2.2] with $(h'' + c^2/2)$ from the differential form of [2.9] and adding the dot product of the momentum equation [2.6] with the velocity $\mathbf{c}$, yields

$$\sum_i \rho n\dot{q}_i + \sum_i \rho n_i\mathbf{D}_i\cdot(\mathbf{c} - \mathbf{u}_i) = \rho(1-y)\left(\frac{Dh''}{Dt} - \frac{1}{\rho''}\frac{Dp}{Dt}\right) + \sum_i \rho n_i\dot{m}_i\left[(\hat{h}_G - h'') + \frac{1}{2}(\mathbf{c} - \mathbf{c}_G)^2\right]. \qquad [2.10]$$

(The condition $\rho/\rho_i \ll 1$, is applicable unless high pressure conditions are being considered.)

Repeating the procedure for droplets of the ith group we have

$$-\int_v n_i\dot{q}_i\rho\,\mathrm{d}v + \int_v n_i\mathbf{D}_i\cdot\mathbf{u}_i\rho\,\mathrm{d}v = \frac{\partial}{\partial t}\int_v \rho y_i\left(e_i + \frac{u_i^2}{2}\right)\mathrm{d}v$$
$$+ \int_s \rho y\left(h_i + \frac{u_i^2}{2}\right)(\mathbf{u}_i\cdot\mathrm{d}\mathbf{s}) - \int_v n_i\dot{m}_i\left(\hat{h}_G + \frac{c_G^2}{2}\right)\rho\,\mathrm{d}v \qquad [2.11]$$

and the differential form

$$-\rho n_i\dot{q}_i = \rho y_i\left(\frac{Dh_i}{Dt_i} - \frac{1}{\rho_i}\frac{Dp}{Dt}\right) - \rho n_i\dot{m}_i\left[(\hat{h}_G - h_i) + \frac{1}{2}(\mathbf{c}_G - \mathbf{u}_i)^2\right]. \qquad [2.12]$$

The group $(\hat{h}_G - h_i)$ is the latent heat of phase change. In a non-equilibrium process some of the total heat transfer is accounted for by the first r.h.s. term. The ratio of this term and the heat transfer varies directly with the expansion rate and inversely as the square of the droplet radius. The term should not be neglected without careful consideration of the particular case concerned.

The flow equations are not restricted to any particular kind of vapour or pressure regime until a specific equation of state is adopted and the relationship between mixture and component densities decided via [1.3]. In order to form a determinate equation set, we may take any two of the separate phase and mixture equations for each of the conservation laws. The differential continuity, momentum, and energy equations for the gas, [2.2], [2.6], [2.10] and the corresponding $3n$ droplet group [2.4], [2.8], [2.12], together with an equation of state, total $(3n+4)$ equations. The dependent variables are taken to be ρ'', c, $\mathbf{u}_i$, p, T, y_i, m_i, T_i. Other variables, e.g. n_i, r_i, h'', etc. can be considered as auxiliary. For a condensing steam flow with fine-droplets it is convenient to follow Gyarmathy (1962) and set the droplet temperature according to the surrounding gas temperature and pressure through the relationship

$$T_i = T_s - (T_s - T'')r^*/r_i \qquad [2.13]$$

where the critical radius

$$r^* = \frac{2\sigma}{\rho_i h_{LG}} \bigg/ \ln\left(\frac{T_s}{T''}\right).$$

Kirillov (1970) gives

$$h_i = h_L - C_L(T_s - T_i) + 3a/\rho_i r_i \qquad \hat{h}_G = h_G - C_p(T_s - T_i)$$

for the droplet and transfer enthalpies respectively when the droplet surface tension is expressed as $\sigma_i = a - bT_i$. T_s and h_L are the 'flat surface' saturation temperature and enthalpy, and the additional terms represent corrections for capilliary effects which are significant for very small droplet sizes. Prescribing the droplet temperature via [2.13] reduces the number of dependent variables to the required $(3n+4)$ and we have then a determinate equation set. An alternative approach is to retain T_i as a dependent variable and introduce an additional droplet growth law. In the free molecule regime ($\mathrm{Kn} \gg 1$), the arguments of kinetic theory lead to the Hertz-Knudsen formula for droplet growth rate

$$\dot{m}_i = \alpha_c r_i^2 \sqrt{\left(\frac{8\pi}{R}\right)}\left(\frac{p}{\sqrt{T''}} - \frac{\hat{p}_s(T_i, r_i)}{\sqrt{T_i}}\right) \qquad [2.14]$$

where $\hat{p}_s$ is the saturation pressure for a droplet of temperature T_i and radius r_i, and α_c is a condensation coefficient

$$\hat{p}_s = p_s(T_i)\,\mathrm{e}^{(2\sigma/\rho_i r_i R T_i)}.$$

The heat transfer rate from a single droplet is written

$$\dot{q}_i = 4\pi r_i^2 \alpha_i (T_i - T'')$$

and slightly adapting Gyarmathy's (1962) expression for heat transfer coefficient to include forced convection effects gives

$$\alpha_i = \left(\frac{k}{2r_i}\right) \frac{\mathrm{Nu}_0}{\left(1 + \frac{\sqrt{(8\pi)}}{1.5}\frac{\gamma}{\gamma+1}\left(\frac{\mathrm{Kn}}{\mathrm{Pr}}\right)\mathrm{Nu}_0\right)} \qquad [2.15]$$

where k is the gas thermal conductivity and Kn the droplet Knudsen number. Nu_0 is the droplet Nusselt number corrected for any relative motion with respect to the gas. From Deich & Fillipov (1968)

$$\mathrm{Nu}_0 = 2 + 0.03\mathrm{Pr}^{0.33}\,\mathrm{Re}^{0.54} + 0.35\,\mathrm{Pr}^{0.356}\,\mathrm{Re}^{0.58}$$

where Re is the relative flow Reynolds number and Pr the Prandtl number for the gas. Gyarmathy (1962) has shown that taking [2.13] in conjunction with [2.15] is a good approximation for the Hertz–Knudsen formula. This procedure has the advantage of smoothly linking the free molecule and continuum regimes. The transfer formulae remain a point of some contention. Puzyrewski & Studzinski (1980) have proposed a droplet growth model which uses a refined form of the Hertz–Knudsen formula and retains droplet temperature as a dependent variable. An intermediate temperature T_i' is defined where the Hertz–Knudsen free molecule model applies across a temperature discontinuity $(T_i - T_i')$ at the droplet boundary. Conventional continuum conduction formulae are assumed to apply between the droplet surface and the far field gas at temperature T''.

From kinetic theory the temperature discontinuity is given as

$$T_i - T_i' = \beta\lambda\left(\frac{\partial T}{\partial r}\right)_{r=r_i}$$

where β is a coefficient of order unity, and λ the gas mean free path.

For heat conduction with spherical symmetry

$$\dot{q} = 4\pi r^2 k\frac{\partial T}{\partial r}$$

Integrating from $r = r_i$ to $r = \infty$, and allowing for relative motion with respect to the surrounding gas

$$\dot{q}_i = 4\pi r_i^2\frac{k}{2r_i}\mathrm{Nu}_0(T_i' - T'')$$

hence

$$T_i' = \frac{T_i + \beta\dfrac{\lambda}{r_i}\dfrac{\mathrm{Nu}_0}{2}T''}{1 + \beta\dfrac{\lambda}{r_i}\dfrac{\mathrm{Nu}_0}{2}} \qquad [2.16]$$

Eliminating T_i'

$$\dot{q} = 4\pi r_i^2\alpha_i(T_i - T'')$$

where

$$\alpha_i = \frac{k}{2r_i}\frac{\mathrm{Nu}_0}{\left(1 + \beta\dfrac{\lambda}{r_i}\dfrac{\mathrm{Nu}_0}{2}\right)}.$$

Following Puzyrewski & Studzinski we may equate the expressions for continuum and free molecule heat transfer to obtain the condition

$$\beta = \frac{(2-\alpha_c)}{2(\alpha_c + \alpha_a(1-\alpha_c))}\frac{k\surd(2\pi RT_i')}{\rho C_p\lambda} \qquad [2.17]$$

Taking

$$\lambda = \frac{1.5\mu''}{\rho''\surd(RT_i')} \quad \text{and} \quad \alpha_c \simeq 1$$

$$\alpha_i = \frac{k}{2r_i}\frac{\mathrm{Nu}_0}{\left[1 + \left(\dfrac{2-\alpha_c}{2\alpha_c}\right)\dfrac{\surd(8\pi)}{1.5}\dfrac{\mathrm{Kn}}{\mathrm{Pr}}(0.5\,\mathrm{Nu}_0)\right]}. \qquad [2.18]$$

It is interesting to note that this is identical in form with Gyarmathy's expression for the heat transfer coefficient [2.15]. The group $(\gamma/(\gamma+1))$ has been replaced by 0.5. If one wishes to take account of Knudsen's effusion effect for energy transfer through small surfaces then C_p is replaced by $(C_p - R/2)$ in [2.17] and the group $(\gamma/(\gamma+1))$ is obtained. The latent heat in [2.12] should be correspondingly modified.

The group $(2\alpha_c/(2-\alpha_c))$ results from the Schrage correction in the modified Hertz–Knudsen formula for droplet growth rate

$$\dot{m}_i = \left(\frac{2\alpha_c}{2-\alpha_c}\right)\sqrt{\left(\frac{8\pi}{R}\right)}\left(\frac{p}{\sqrt{T_i'}} - \frac{\hat{p}_s(T_i, r_i)}{\sqrt{T_i}}\right). \qquad [2.19]$$

The associated relaxation time

$$\tau_3 = \frac{(2-\alpha_c)}{2\alpha_c}\frac{\sqrt{(2\pi)}}{3}\rho_i r_i \frac{\sqrt{(RT_i)}}{p}$$

proves to be two orders of magnitude smaller than the thermal relaxation time τ_1. Equations [2.12] and [2.19] together constitute the original Puzyrewski & Studzinski model, but the possibility of discarding [2.19] and using Gyarmathys equation [2.13] for droplet temperature in conjunction with [2.18] for the heat transfer coefficient suggests itself as a likely accurate approximation.

This abbreviation of the model avoids the computational restrictions imposed by the relaxation time τ_3.

3. NON EQUILIBRIUM LOSS PROCESSES

Before moving on to particular applications of the flow equations it is instructive to consider the corresponding expressions for entropy increase. This provides a means to compare internal losses resulting from non equilibrium exchanges of heat and momentum between the phases with the conventional aerodynamic losses associated with boundary layers, shock waves, etc. which arise in transonic steam turbine blading.

In steady flow the gradient of entropy flux is given by

$$\nabla \cdot \left[\rho(1-y)\mathbf{c}s'' + \sum_i \rho y_i \mathbf{u}_i s_i\right] = \rho(1-y)\mathbf{c} \cdot \nabla s'' + \sum_i \rho y_i \mathbf{u}_i \cdot \nabla s_i - \sum_i \rho n_i \dot{m}_i (s'' - s_i) \qquad [3.1]$$

since from continuity

$$\nabla(\rho(1-y)\mathbf{c}) = -\sum_i \nabla(\rho y_i \mathbf{u}_i) = -\sum_i \rho n_i \dot{m}_i \,.$$

Making the substitutions

$$T''\frac{Ds''}{Dt} = \frac{Dh''}{Dt} - \frac{1}{\rho''}\frac{Dp}{Dt} \qquad T_i\frac{Ds_i}{Dt_i} = \frac{Dh_i}{Dt_i} = \frac{Dh_i}{Dt_i} - \frac{1}{\rho_i}\frac{Dp}{Dt}$$

in the energy equations [2.10] and [2.12] we have

$$(1-y)T''\mathbf{c} \cdot \nabla s'' = \sum_i n_i \dot{q}_i + \sum_i n_i D_i \cdot (\mathbf{c} - u_i) - \sum_i n_i \dot{m}_i \left[(h_G - h'') + \frac{1}{2}(\mathbf{c} - \mathbf{c}_G)^2\right]$$

$$y_i T_i \mathbf{u}_i \cdot \nabla s_i = -n\dot{q}_i + \sum_i n_i \dot{m}_i \left[(h_G - h_i) + \frac{1}{2}(\mathbf{c}_G - \mathbf{u}_i)^2\right]. \qquad [3.2]$$

For the condensing case $\mathbf{c}_G = \mathbf{c}$ and the entropy flux gradient is

$$\sum_i \rho n_i \left[(q_i - \dot{m}_i(h_G - h_i)) \left(\frac{1}{T''} - \frac{1}{T_i} \right) + \dot{m}_i \left(\frac{(h'' - h_i)}{T''} - (s'' - s_i) \right) + \dot{m}_i \frac{1}{2} \frac{(\mathbf{c} - \mathbf{u}_i)^2}{T_i} + \frac{\mathbf{D}_i \cdot (\mathbf{c} - \mathbf{u}_i)}{T''} \right]. \qquad [3.3]$$

For the purpose of illustration assume a perfect gas and saturation droplet states, then $T_i = T_s$, $h_i = h_L$, $s_i = s_L$, $T_s s_{LG} = h_{LG}$, and

$$h'' - h_G = C_p(T'' - T_s) \qquad s'' - s_G = C_p \ln \frac{T''}{T_s}.$$

Put $\Delta T = T_s - T''$, $\Delta u = \mathbf{c} - \mathbf{u}$, and take $q = \dot{m}_i h_{LG}$ (which for small droplets is a good approximation) then the first group of terms in expression [3.3] vanish, we have finally,

$$\sum_i \rho n_i \dot{m}_i C_p \chi$$

for the thermodynamic component of entropy change where

$$\chi = \left[\frac{\left(\frac{h_{LG}}{C_p T_s} - 1 \right) \frac{\Delta T}{T_s}}{\left(1 - \frac{\Delta T}{T_s} \right)} - \ln \left(1 - \frac{\Delta T}{T_s} \right) \right]. \qquad [3.4]$$

The behaviour of the function χ is shown in figure 5. $\dot{m}_i$ is positive or negative depending on whether the gas is supercooled ($\Delta T > 0$) or superheated ($\Delta T < 0$) hence the thermodynamic entropy change is always positive or zero as required by the second law. The same is true of the entropy change due to relative motion between the phases which varies as the square of their velocity difference Δu.

We may define a generalised loss coefficient to include non-equilibrium effects by analogy with the traditional Markov loss coefficient for single phase flow.

Markov gives $\xi = -(T\,\mathrm{d}s/\mathrm{d}h_s)$ where $\mathrm{d}h_s = (1/\rho)\,\mathrm{d}p$ the suffix s denoting an isentropic enthalpy change. Define ξ

$$\xi = \frac{T''\nabla \cdot \left[\rho(1 - y)\mathbf{c}s'' + \sum_i \rho y_i \mathbf{u}_i s_i \right]}{-\rho\mathbf{c} \cdot \left(\frac{1}{\rho} \nabla p \right)} \qquad [3.5]$$

as a ratio of the divergences of the entropy and isentropic enthalpy fluxes then this new loss coefficient reduces to the Markov definition in the case of equilibrium flow.

In steady flow $-\mathbf{c} \cdot \nabla p$ is the local expansion rate $-\dot{p}$ of the process so for thermodynamic loss

$$\xi_1 = C_p T'' \sum_i \rho n_i \dot{m}_i \chi / (-\dot{p}) \qquad [3.6]$$

and for kinematic loss

$$\xi_2 = \sum_i \rho \left[n_i \mathbf{D}_i \cdot (\mathbf{c} - \mathbf{u}_i) + n_i \dot{m}_i \frac{1}{2} (\mathbf{c} - \mathbf{u}_i)^2 \frac{T''}{T_i} \right] \Big/ (-\dot{p}). \qquad [3.7]$$

The droplet growth rate and drag can be calculated with the aid of the relaxation times defined in section 5.

$$n_i\dot{m}_i = \frac{y_i C_p \Delta T}{C_m h_{LG} \tau_1} \qquad n_i D_i = y_i \frac{\Delta u}{\tau_2}$$

Taking a supercooling ΔT of 30°C and plausible values for slip speed Δu in [3.6] and [3.7] for 3% monodispersions of various droplet sizes in steam at 0.1 bar, gives the following table of loss coefficients

r μm	τ_1 μs	τ_2 μs	Δu m/s	ξ_1	ξ_2
0.01	7.55	0.17	0	0.476	—
0.1	82.6	1.92	10	0.0435	0.0011
1.0	1530.0	38.1	30	0.0023	0.0005

The coefficients have been based on an expansion rate of 10^3 bar/sec. Konorski quotes rates between 10^2 and 10^3 bar/sec as typical of L.P. steam turbines. The highest thermodynamic losses occur for the combination of very small droplets and deep supersaturation which arises in practice following nucleation when fresh droplets grow rapidly. It is doubtful that the kinematic losses ever exceed the order of 0.1% whilst instantaneous thermodynamic losses of order 10% can occur. The instantaneous loss coefficients provide only limited information since deep supersaturation occurs only transiently in steam turbines and to arrive at the final overall loss requires detailed computation of the flow history.

4. ONE DIMENSIONAL CHOKING IN LOW PRESSURE WET STEAM

The equations for steady one dimensional flow through a duct of cross sectional area $A(x)$ are found by making the appropriate reductions to the more general equations given in section 2. The algebra is simplified by choosing separate phase equations. For the sake of conciseness the equations are given for a monodispersion of droplets. The gas phase is assumed calorically perfect so we have

$$C_p - C_v = R \quad \text{and} \quad h'' = C_p T'' + h_0.$$

From the gas and liquid continuity equations [2.1] and [2.3]

$$\frac{1}{\rho''}\frac{d\rho''}{dx} + \frac{1}{c}\frac{dc}{dx} + \frac{nu}{c(1-y)}\frac{dm}{dx} = -\frac{1}{A}\frac{dA}{dx}$$

$$\frac{1}{\rho''}\frac{d\rho''}{dx} + \frac{1}{u}\frac{du}{dx} + \frac{1}{y(1-y)}\frac{dy}{dx} - \frac{1}{m}\frac{dm}{dx} = -\frac{1}{A}\frac{dA}{dx}.$$

The momentum equations [2.6] and [2.8] reduce to

$$(1-y)c\frac{dc}{dx} + \frac{(1-y)}{\rho''}\frac{dp}{dx} = -nD$$

$$yu\frac{du}{dx} - (c-u)nu\frac{dm}{dx} = nD$$

and from the energy equations [2.10] and [2.12] we have

$$(1-y)C_p c\frac{dT''}{dt} - \frac{(1-y)c}{\rho''}\frac{dp}{dx} + C_p(T'-T'')nu\frac{dm}{dx} = n\dot{q} + nD(c-u)$$

$$\frac{yC_L RT'^2}{ph_{LG}}u\frac{dp}{dx} - \left[h_{LG} + \frac{1}{2}(c-u)^2\right]nu\frac{dm}{dx} = -n\dot{q}$$

where the drag

$$D = \frac{3\pi\mu'' d(c-u)}{(1+2.53\mathrm{Kn})}\left[1 + \left(\frac{|c-u|\rho'' d}{60\mu''}\right)^{0.5556}\right]^{1.8}$$

is calculated on a quasi-steady basis.

These equations together with the equation of state in differential form

$$\frac{1}{\rho''}\frac{\mathrm{d}\rho''}{\mathrm{d}x} - \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{T''}\frac{\mathrm{d}T''}{\mathrm{d}x} = 0$$

may be expressed in matrix notation

$$[A]\frac{\mathrm{d}}{\mathrm{d}x}\{v\} = \{B\} \qquad [4.1]$$

where $[A]$ is the square matrix of derivative coefficients.

$\{v\}$ and $\{B\}$ are column vectors of dependent variables and "source" terms respectively.

$$\{v\}^T = (\rho'', c, u, p, T'', y, m)$$

$$\{B\}^T = \left(-\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}, -\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x}, -nD, nD, n\dot{q} + nD(c-u), -n\dot{q}, 0\right).$$

To solve this system of equations we require explicit equations for the derivatives, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{v\} = \frac{\mathrm{adj}[A]\{B\}}{\det[A]}.$$

The equations are singular if $\det[A] = 0$ which is the condition for choked flow. A procedure given by Davidson (1974) for removing the singularity has been employed. The independent variable change $x = |\det[A]|z$ is made giving

$$\frac{\mathrm{d}}{\mathrm{d}z}\{v\} = \mathrm{sgn}\,(\det[A])\mathrm{adj}\,[A]\{B\}$$

which can be integrated in the region of the singularity. For cases of choked flow it is necessary to select an initial value for the gas velocity c such that $\mathrm{adj}[A]\{B\}$ and $\det[A]$ reach zero simultaneously, since only then can the derivatives $(\mathrm{d}/\mathrm{d}x)\{v\}$ remain finite at the singularity. A computer program SLIPFLO was written to integrate the equations numerically using a fourth order Runge–Kutta method.

4.1 *The choking flow condition and the effect of simplifications in the flow model*

The singularity criterion, $\det[A] = 0$, for [4.1] gives

$$c^* = a''\Big/\left[1 + \frac{y}{(1-y)}(\gamma-1)\left(\frac{C_pT'}{h_{LG} + \frac{1}{2}(c-u)^2}\right)\left(\frac{C_LT'}{h_{LG}}\right)\frac{u}{c}\frac{T'}{T''}\right]^{1/2}. \qquad [4.2]$$

The singular velocity c^* is slightly less than the acoustic speed of the gas phase alone $a'' = \sqrt{(\gamma RT'')}$, and greater than the equilibrium acoustic speed a_e, given by [5.4]. It can be shown from the compatibility condition, $\mathrm{adj}[A]\{B\} = 0$, that the singularity condition occurs a small distance downstream of the geometric throat in a condensing flow (e.g. figure 1). It is necessary to compute the preceeding flow history to accurately determine the choking mass flow, G.

$$G = \rho^* A^*[(1 - y^*)c^* + y^* u^*].$$

The expressions for the singular velocity given by Gyarmathy & Moore (see Moore & Sieverding 1976) differ from [4.2], chiefly by a factor of $(1 - y)^{-1/2}$. This is a consequence of their use of the homogeneous flow model in which the droplets and gas are assigned a common velocity. The singularity condition is as Gyarmathy remarks, formulation dependent. If for example the droplet temperature T' is treated as a dependent variable and a droplet growth law is used, the simpler condition $c^* = a^*$ ensues. The complexity of [4.2] is then a consequence of simplifying the droplet temperature dynamics. The numerical effect on c^* is almost negligible, typically less than 0.1%, and the computational advantages are considerable.

The situation with regard to the homogeneous flow assumption is more troubled since it causes reductions in c^* of several percent, however the complexity of computing separate droplet motion in two or more dimensions for several droplet groups provides a strong incentive to seek other reductions. For small droplets it is very tempting to discard

(i) All terms involving velocity differences between the phases,

(ii) The equations of continuity and momentum for the droplets when the droplet motion itself is not of interest.

The approximation entailed in setting $\mathbf{u}_i = \mathbf{c}$ depends on whether mixture or separate phase equations are used. In the mixture equations it constitutes the homogeneous flow assumption and implies an artificially high interphase drag, which slightly reduces the choking mass flow. A false dependance of the singularity condition and acoustic speed on wetness fraction also results. The approximation may however be made in the separate phase equations without distorting the transonic behaviour. This amounts to neglecting the interphase drag and associated kinematic losses. The choking flow is hence overestimated (see section 4.3). For very small droplets ($r < 0.1\ \mu$m) very short inertial relaxation times apply (e.g. $\tau_2 < 1\ \mu$sec) and it becomes time consuming to compute separate droplet motion from its differential momentum equation, because of the corresponding restrictions on calculation step length.

4.2 *Approximations for choking mass flow*

As the expansion rates in L.P. turbine blade passages are quite rapid it is reasonable to examine how closely the flow conditions approximate to frozen flow. The term "frozen flow" refers to a hypothetical limiting case in which no heat or mass transfer takes place between the phases. In frozen flow the gas behaves as if it were in isolation and we have the usual gas dynamic expressions for the singular state properties p^*, T^*, ρ^* in terms of stagnation properties p''_0, T''_0, ρ''_0 at duct entry. The constants C_p, R and specific heat ratio $\gamma(\simeq 1.32)$ for low pressure steam are chosen to fit the gas properties in the region of the saturation line. The singularity condition $c^* = \sqrt{(\gamma R T^*)}$ occurs at the geometric throat, thus for the frozen mass flow

$$G_f = (\rho'' A_t c)^* + G'$$

Now the proportion of the overall mass flow contributed by the droplets remains unchanged between the inlet and the throat so, $G' = y_1 G_f$ and

$$G_f = \frac{(\rho'' A_t c)^*}{(1 - y_1)} = A_t \sqrt{\left[\gamma p''_0 \rho''_0 \left(\frac{2}{\gamma + 1}\right)^{(\gamma+1)/(\gamma-1)}\right]} \Big/ (1 - y_1). \qquad [4.3]$$

The equilibrium value for choking mass flow, G_e can be obtained to reasonable accuracy by assuming an effective expansion exponent γ_e of 1.12 for the mixture as a whole. This leads to the same form of expression as in [4.3] but with slightly different stagnation properties and without the factor $(1-y_1)^{-1}$.

The ratio $(\mu_f = G_f/G_e)$ of frozen and equilibrium choking flows is plotted against inlet wetness in figure 4. It has been assumed that the inlet flow is initially in equilibrium. The supercooling developed in a frozen flow expansion to the throat and the consequent value of the flow coefficient μ_L decrease as the inlet Mach number approaches unity. For an inlet Mach number of 0.5 and a wetness of 3%, we have for example from figure 4 that the frozen flow exceeds the equilibrium calculation by some 6%. The actual choking flow will be somewhere between the frozen and equilibrium limits for wet inlet conditions since condensation on existing droplets allows a partial restoration of equilibrium.

4.3 *Some computed examples of choked wet steam flows*

In order to assess the extent of non equilibrium effects on the choking process in the final low pressure stages of steam turbines various flows in a convergent divergent duct were computed using SLIPFLO. The duct, intended to be representative of a blade passage, was a shortened version of one investigated by Bakhtar & Young (1978). It had a contraction ratio of 1.5 and measured 5.08 cm from entry to throat, the included divergence angle of the walls downstream of the throat was 2°. Monodispersions with initial droplet radii ranging from 0.01 to 5 μm, and wetness fractions of 0.03, 0.05, and 0.10 were considered. Equilibrium entry conditions at a pressure of 0.1 bar were assumed. Figure 1 shows the Mach number and

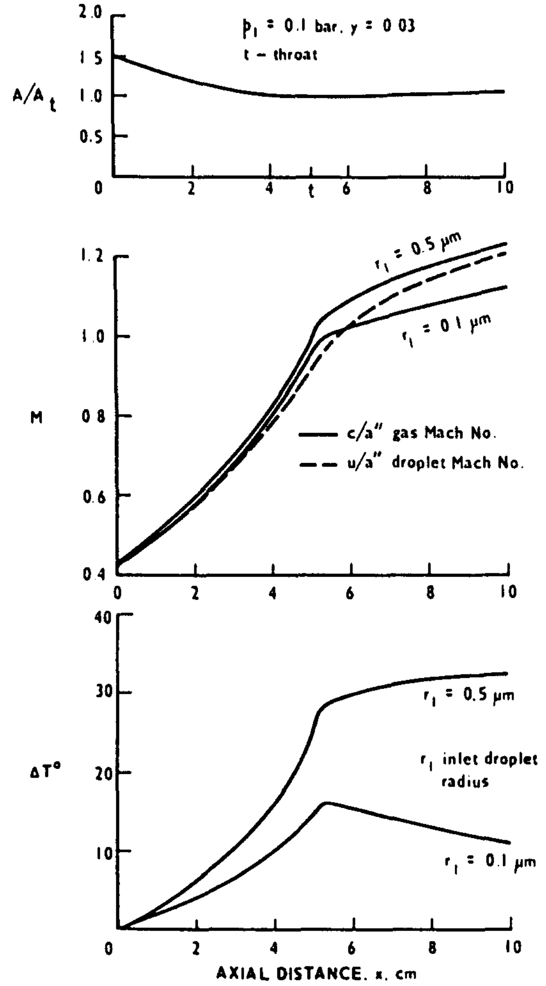

Figure 1. Area, Mach number and supercooling distributions for the test duct.

supersaturation distributions along the duct for an inlet wetness of 3% for two different droplet sizes. The throat Mach number is less than unity and this departure from pure gas behaviour is most pronounced for the smaller droplet sizes. The 0.1 μm case gave a throat Mach number of 0.972 and a singularity throat separation of about 3 mm. The slip between the phases reached approximately 30 m/sec for the 0.5 μm droplets and remained less than 10 m/sec for droplets of less than 0.1 μm radius. The larger droplet flows having less surface area available for interphase heat transfer, exhibit the deepest supersaturation. The large supercooling (>30°C) for the 0.5 μm droplet case would be relieved by a secondary nucleation in the supersonic zone not shown in the figure. Bakhtar & Young have described the effect of the duct divergence angle (i.e. the supersonic expansion rate) on the choking behaviour.

Figure 2 shows choking mass flows computed for the test duct, expressed as a ratio of the "theoretical frozen" choking flow. The frozen choking flow is independent of the duct geometry and forms an upper limiting asymptote which is approached by increasing the droplet size or reducing the wetness fraction. Converse changes in droplet size and wetness fraction give flows which are nearer equilibrium values. (Equilibrium flows corresponding to the three wetness fractions, marked E_{03}, E_{05} and E_{10} in figure 2, have been calculated by the approximate effective exponent method, e.g. figure 4.) The duct expansion rate and inlet pressure level also influence the choking flow. A duct with a faster expansion rate would give curves tending more quickly to the frozen flow asymptote, whilst a higher inlet pressure (1.0 bar say) improves the free molecule heat transfer for a given droplet size and produces a slight shift towards equilibrium behaviour.

Figure 3 shows the choking mass flow variation obtained with revised Knudsen number weighting in the expression for heat transfer coefficient (i.e. [2.18]). Comparison with figure 2 reveals that the reweighted calculation gives a more rapid approach towards equilibrium behaviour with reducing droplet size whilst the large droplet results are unchanged. The full Puzyrewski & Studzinski model was computed for just two cases (0.3 and 5.0 μm droplets) and the results for choking mass flow and development of supersaturation were difficult to distinguish from the calculation with revised Knudsen weighting. The computation was however at least an order of magnitude slower.

The chained curves in figure 2 show flows computed with separate phase equations but velocity different terms neglected as discussed earlier (reductions (i) and (ii)). The smaller

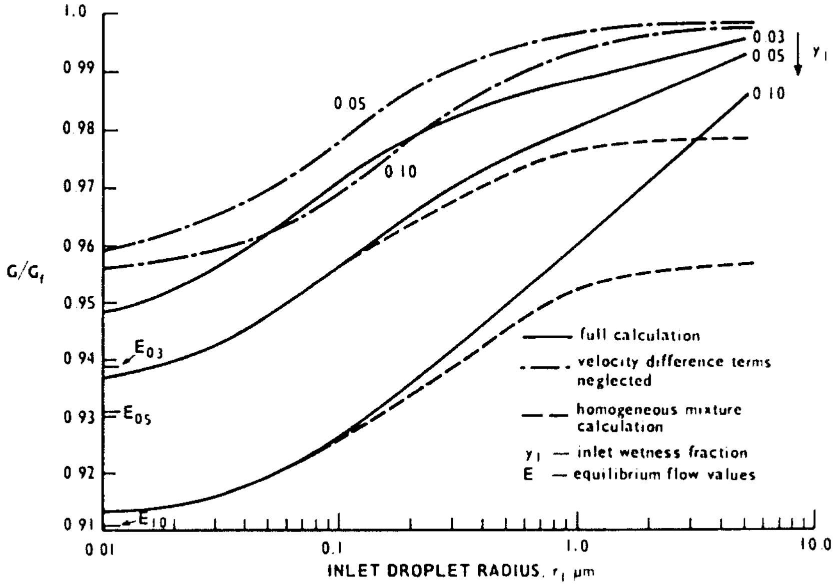

Figure 2. Choking mass flows for various duct inlet conditions, with droplet growth calculated from [2.16] and [2.21].

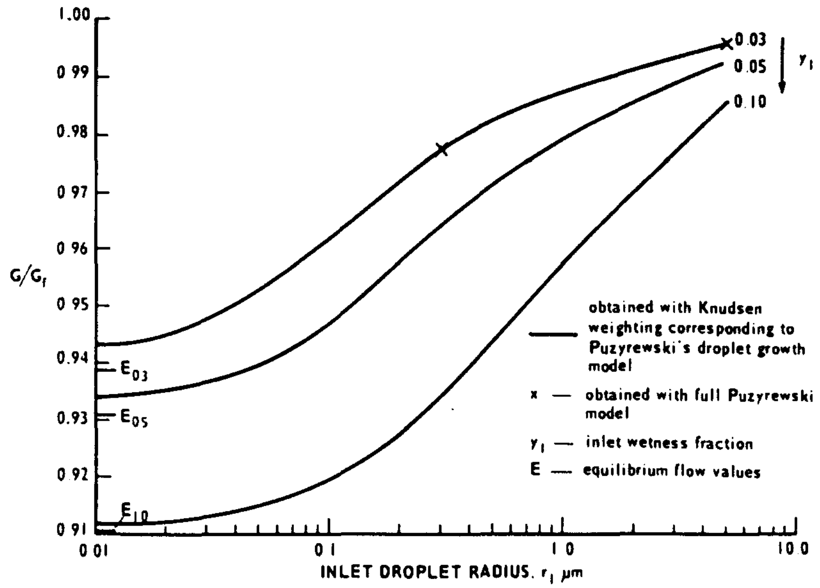

Figure 3. Choking flows obtained with revised droplet growth formulae.

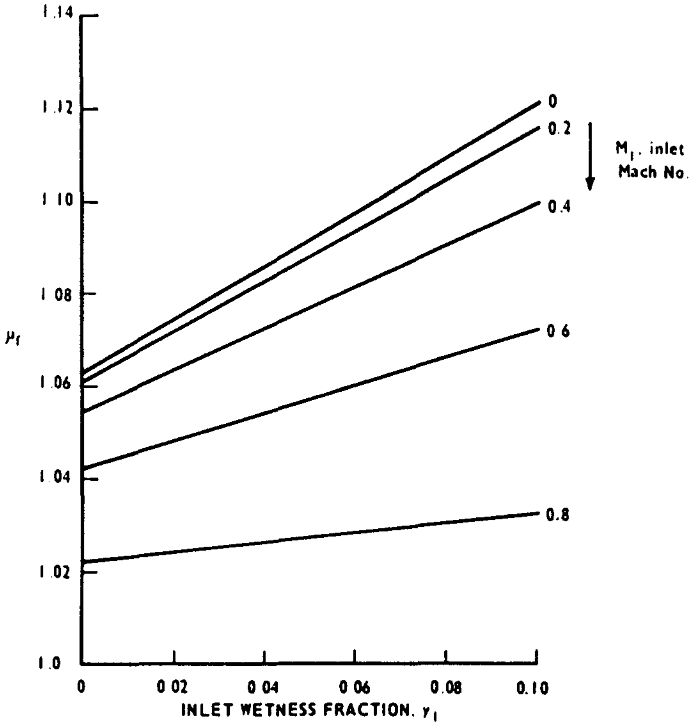

Figure 4. Non equilibrium flow coefficient, $\mu_f = G_f/G_e$

$$\mu_f = \sqrt{\left(\frac{\gamma}{\gamma_e(1-y_1)}\right)}\frac{\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_1^2\right)\right]^{(\gamma+1)/[2(\gamma-1)]}}{\left[\frac{2}{\gamma_e+1}\left(1+\frac{\gamma_e-1}{2}M_e^2\right)\right]^{(\gamma_e+1)/[2(\gamma_e-1)]}}$$

where

$$\frac{M_e}{M_1} = \sqrt{\left(\frac{\gamma}{\gamma_e(1-y_1)}\right)}.$$

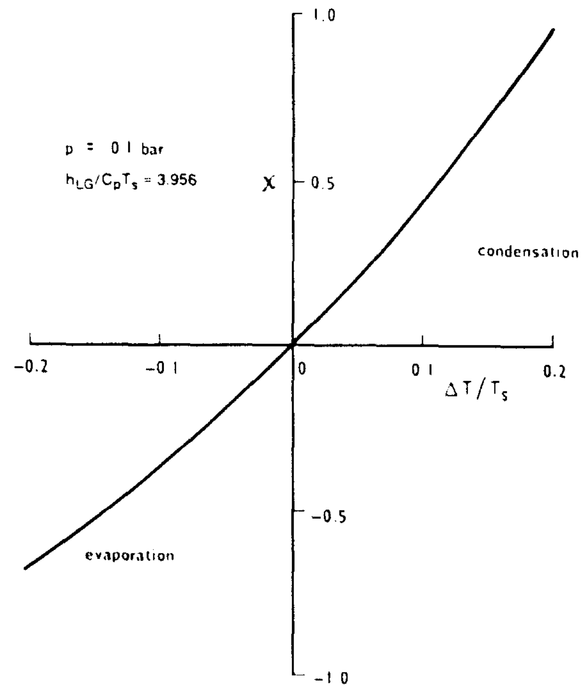

Figure 5. Function describing the thermodynamic component of entropy change.

droplet flows exceed those of the full calculation by about 2% for the 0.05 wetness case. The approximation is best for low wetness fractions and large droplets.

The dashed curves show flows computed with the homogeneous flow mixture equations. They agree well with the full calculation for small droplets but do not approach the single frozen flow asymptote with increasing droplet size. This formulation also has artificial acoustic properties and should be avoided if possible. The approximation

$$\mathbf{u}_i = \mathbf{c} - \tau_2 \frac{D\mathbf{c}}{Dt}$$

obtained by neglecting the derivative of the relative velocity $(\mathbf{c} - \mathbf{u}_i)$ will often give the droplet velocity field to sufficient accuracy.

5. ACOUSTIC PROPAGATION IN LOW PRESSURE WET STEAM

To understand the propagation of small waves in a two phase medium one only need consider linearised forms of the flow equations. Accordingly we define small perturbations ($\tilde{p}$, $\tilde{c}$, etc.) about mean values of the flow variables which are designated with a zero suffix, i.e. put $p'' = p_0 + \tilde{p}, \ldots$ etc. where $(\tilde{p}/p_0) \ll 1$. The velocities are taken relative to the mean fluid motion and the temperature perturbations are defined with respect to the saturation condition at the mean pressure (i.e. $c_0 = 0$, $T''_0 = T_s(p_0)$). Consider one dimensional wave propagation with the direction of propagation along the z axis.

In terms of perturbation quantities, neglecting second order terms, [2.2], [2.4], [2.6], [2.8], [2.10] and [2.12] reduce to:

$$\begin{gathered}
\frac{\partial\tilde{\rho}}{\partial t}+\rho_0\frac{\partial\tilde{c}}{\partial z}=-\rho_0\frac{C_m}{m_0}\frac{\partial\tilde{m}}{\partial t}\\
C_m\frac{\partial\tilde{\rho}}{\partial t}+\rho_0C_m\frac{\partial\tilde{u}}{\partial z}+\frac{\rho_0}{(1-y_0)^2}\frac{\partial\tilde{y}}{\partial t}=\rho_0\frac{C_m}{m_0}\frac{\partial\tilde{m}}{\partial t}\\
\rho_0\frac{\partial\tilde{c}}{\partial t}=-\frac{\partial\tilde{p}}{\partial z}-\rho_0\frac{C_m}{\tau_2}(\tilde{c}-\tilde{u})\qquad\frac{\partial\tilde{u}}{\partial t}=\frac{1}{\tau_2}(\tilde{c}-\tilde{u})\\
\frac{\partial\tilde{T}}{\partial t}-\frac{1}{C_p\rho_0}\frac{\partial\tilde{p}}{\partial t}=\frac{1}{\tau_1}(\tilde{T}'-\tilde{T}).\\
C_L\frac{\partial\tilde{T}'}{\partial t}-\frac{h_{LG}}{m_0}\frac{\partial\tilde{m}}{\partial t}=-\frac{C_p}{C_m\tau_1}(\tilde{T}'-\tilde{T})
\end{gathered}\tag{5.1}$$

C_m is the group $y/(1-y)$. The thermal and kinematic relaxation times τ_1 and τ_2 respectively are identified from the form of the differential equations for the gas temperature and droplet velocity.

$$\tau_1=\frac{C_p\rho' r_0^2}{3C_mk}\left(1+3.8\frac{\mathrm{Kn}}{\mathrm{Pr}}\right)\quad\tau_2=\frac{2\rho' r_0^2}{9\mu''}(1+2.53\,\mathrm{Kn})$$

For low pressure steam the linearised equation of state is

$$\frac{1}{p_0}\frac{\partial\tilde{p}}{\partial t}-\frac{1}{\rho_0}\frac{\partial\tilde{\rho}}{\partial t}-\frac{1}{T_0}\frac{\partial\tilde{T}}{\partial t}=0$$

and from the Clausius Clapeyron equation the saturation temperature perturbation $\tilde{T}_s$ satisfies

$$\frac{\partial\tilde{T}_s}{\partial t}=\frac{RT_0^2}{h_{LG}p_0}\frac{\partial\tilde{p}}{\partial t}$$

Again the option exists of setting the droplet temperature (i.e. taking $\tilde{T}'=T_s$) or treating it as a free variable and introducing the linearised Hertz Knudsen droplet growth law.

Now we assume travelling harmonic wave solutions of the form

$$\tilde{p}=\beta\,\mathrm{e}^{i(\omega t-\kappa z)}=\beta\,\mathrm{e}^{-\lambda z}\,\mathrm{e}^{i\omega(t-z/a)}$$

where β is the wave amplitude, which is in general complex and describes the phase of the wave; a is the speed of wave propagation (i.e. the acoustic speed); λ is the wave attenuation coefficient; a and λ are given by the real and imaginary parts of the wave number κ i.e. $a=\omega/\mathrm{Re}(\kappa)$ and $\lambda=-\mathrm{Im}(\kappa)$.

Substituting harmonic solutions for ($\tilde{\rho}$, $\tilde{c}$, $\tilde{u}$, $\tilde{p}$, etc.) just consists of replacing the derivative operators $\partial/\partial t$, $\partial/\partial z$ by $i\omega$ and $-i\kappa$ respectively which leads to an equation system:

$$[A]\{v\}=\{0\}\ \text{where}\ \{v\}^T=(\tilde{\rho},\tilde{c},\tilde{u},\tilde{p},\tilde{T},\tilde{y},\tilde{m})$$

For a non trivial solution the determinant of the coefficient matrix $[A]$ must vanish, and this condition gives the dispersion equation

$$\left(\frac{\kappa}{\omega}\right)^2=\frac{1}{a_e^2}\frac{(1+xi\omega\tau_2)\left(1+\frac{1}{x}\frac{a_e^2}{a_f^2}i\omega\tau_1\right)}{(1+i\omega\tau_1)(1+i\omega\tau_2)}\tag{5.2}$$

where

$$a_f^2 = \frac{C_p T'' p v''}{\left[C_p T'' - p v'' + C_m\left(\frac{C_p T''}{h_{LG}}\right)\left(C_L \frac{RT''^2}{h_{LG}}\right)\right]} \qquad [5.3]$$

and

$$a_e^2 = \frac{p v''(1-y)}{\left[1 - \frac{2RT'}{h_{LG}} + \frac{RT'^2}{h_{LG}} C_p\left(1 + \frac{C_m C_L}{C_p}\right)\right]}. \qquad [5.4]$$

(NB. The mean value notation has been discarded and x now denotes $(1-y)$.)

Separating the real and imaginary parts of [5.2] gives the dispersion and attenuation laws for wave propagation in low pressure wet steam, i.e. the variation of acoustic speed and attenuation coefficient with frequency. The speeds a_f and a_e are the upper and lower dispersion limits corresponding to "infinite" and "zero" frequency wave propagation respectively. For any real non zero frequency the acoustic speed lies between these limits. The lower limit a_e is the equilibrium propagation speed. This may be shown by deriving [5.4] from purely thermodynamic arguments by considering the partial derivative $(\partial p/\partial \rho)_s$ for the mixture as a whole with the assumption of equal phase temperatures. Equations [5.3] and [4.2] for the upper dispersion limit a_f, and the singular velocity in choking flow, are identical within the accuracy implicit in the linearisation. This is to be expected since it guarantees that no pressure signal may propagate upstream of the singularity, whatever its form, which is consistent with the conventional conception of the choking process. The speed a_f relates to propagation under frozen flow conditions which are approached when processes become rapid relative to the relaxation times (i.e. $\omega\tau_2 \gg 1$). If the droplet temperature T' is treated as a free variable $a_f = a''$ is obtained in place of [5.3] but the dispersion law becomes more complex algebraically. It is fairly certain that numerical differences in the dispersion between the two formulations are small. The variations in acoustic speed and attenuation for monodispersions of radii (0.1 and 1.0 μm) in steam of 0.1 bar have been computed from equation (5.2) and found to agree with the results of Petr (1973).†

The gas perturbation velocity $\bar{c}$ and its associated potential function ϕ (i.e. $\bar{c} = -\partial\phi/\partial z$) must satisfy a generalised wave equation of the form

$$\frac{\partial^2\phi}{\partial z^2} - \left(\frac{\kappa}{\omega}\right)^2 \frac{\partial^2\phi}{\partial t^2} = 0 \qquad [5.5]$$

where the wave speed ω/κ is complex.

Substituting for κ/ω from [5.2] and noting that for a unit wave solution $e^{i(\omega t - \kappa z)}$ the operations $\partial/\partial t$, $\partial^2/\partial t^2$, $\partial^2/\partial z^2$ are equivalent to multiplications by $i\omega$, $-\omega^2$ and $-\kappa^2$ respectively, we find that ϕ must satisfy the partial differential equation

$$\tau_1\tau_2\left(\frac{\partial^2\phi}{\partial z^2} - \frac{1}{a_f^2}\frac{\partial^2\phi}{\partial t^2}\right) + (\tau_1+\tau_2)\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial z^2} - \frac{1}{a_f^2}\frac{\partial^2\phi}{\partial t^2}\right) + \left(\frac{\partial^2\phi}{\partial z^2} - \frac{1}{a_e^2}\frac{\partial^2\phi}{\partial z^2}\right)$$
$$+ \left[(1 - x a_f^2/a_e^2)\tau_2 - \frac{y}{x}\tau_1\right]\frac{\partial}{\partial t}\left(\frac{1}{a_f^2}\frac{\partial^2\phi}{\partial t^2}\right) = 0. \qquad [5.6]$$

The above is an 'acoustic' equation for low pressure wet steam. A little re-arrangement of terms

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†There are some typographical errors in the dispersion equations quoted in Petrs paper.

confirms that it is exactly the equation given by Petr, and if we ignore the kinematic relaxation time and concentration of the secondary phase it reduces to the form of Einstein's acoustic dispersion equation for chemically reacting gases (see Wegener 1970). Just as in classical compressible flow theory one may proceed from the acoustic equation to a two or three dimensional potential equation with respect to fixed co-ordinates by making a change of independent variables. This provides a basis for approaching slender body flow problems and Petr (1979) has recently obtained solutions for supersonic wedge flows by finding co-ordinate transformations analogous to the Prandtl–Glauert transformation.

6. CONCLUSIONS

It is necessary to allow distinct velocities for the gas and liquid phases in multiphase flow equations if the correct transonic behaviour of a multiphase medium is to be reproduced. It has been shown that the condition of choking flow occurs when the flow velocity reaches the "frozen" acoustic speed at some cross section in a nozzle flow. This is also the limiting propagation speed for small pressure waves. The expression for "frozen" acoustic speed depends on whether droplet temperature is treated as a free variable in the mathematical model but it may in either event be taken as equal to the acoutic speed of the gas alone for all practical numerical purposes. The choking mass flow varies between limits corresponding to equilibrium and frozen flow behaviour, according to the droplet size, wetness fraction, and the flow expansion rate. The variation is also governed by the heat and mass transfer formulae adopted. The Puzyrewski & Studzinski model for droplet growth (assuming a condensation coefficient of around unity) predicts faster rates of transfer for small droplets than Gyarmathy's model, and consequently approaches equilibrium behaviour in the choking problem more quickly. Gyarmathy's model with revised Knudsen weighting appears to be a very good approximation to Puzyrewski and Studzinski's model and has considerable computational advantages, but further study of droplet growth calculations following nucleation is required to establish whether this simplification of the droplet temperature dynamics remains valid over a wide range of conditions.

In design mass flow calculations for turbine stages it is best to work from the frozen flow limiting values making a downward revision according to the information on expansion rate and wetness available. The internal loss associated with the relative motion between phases is small compared with thermodynamic losses resulting from interphase exchanges of heat and matter.

The equation set for wet vapour flows given in both integral and differential forms is properly determined and consistent in that individual phase equations sum exactly to give the required overall mixture equations, and all the basic physical conservation laws are observed. The equations have been shown to embody the correct physical properties in a number of fundamental problems and should be suitable as a basis for more complex condensing flow problems of concern to the Engineer.

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